A CONSISTENT NONPARAMETRIC TEST OF AFFILIATION IN AUCTION MODELS

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Abstract. In this paper we propose a new nonparametric test of affiliation, a strong form of positive dependence with independence as a special, knife–edge, case. The test is consistent against all alternatives to affiliation, i.e. under the alternative hypothesis (no affiliation), the rejection probability converges to one as the sample size increases to infinity. The test has many uses, including testing the fundamental assumptions of auction theory, differentiating between heterogeneity which is observed to the bidders (but not to the econometrician) and heterogeneity which is unobserved, and testing for collusion. Our test is nonparametric in the sense that it does not require any ex ante distributional assumptions nor indeed a model. It uses empirical distribution functions (EDF), but does require a sample–size dependent input parameter.

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1. Introduction

In this paper we propose a new nonparametric test of affiliation, a strong form of positive dependence with independence as a special, knife–edge, case. The test is consistent against all alternatives to affiliation, i.e. under the alternative hypothesis (no affiliation), the rejection probability converges to one as the sample size increases to infinity. The test has many uses, including testing the fundamental assumptions of auction theory, differentiating between heterogeneity which is observed to the bidders (but not to the econometrician) and heterogeneity which is unobserved, and testing for collusion. Our test is nonparametric in the sense that it does not require any ex ante distributional assumptions nor indeed a model. It uses empirical distribution functions (EDF), but does require a sample–size dependent input parameter.

Milgrom and Weber (1982) established the existence of a pure strategy equilibrium featuring a monotonic bid function in a wide variety of auctions settings. One of their main assumptions is that signals (or values[1]) are affiliated. De Castro (2007) has established that if affiliation is replaced with even a slightly weaker notion of positive dependence,[2] then the existence of a pure strategy monotonic equilibrium is no longer guaranteed. However, as de Castro has shown, the class of cases in which affiliation is satisfied is small relative to the class of cases in which a pure strategy monotonic equilibrium exists. Indeed, affiliation is an extremely strong form of positive dependence, and therefore questionable, but it is not routinely tested in empirical work. Our test allows one to test the affiliation assumption on the raw data, i.e. *before* any modelling assumptions are made.

If the bid function is monotonic, then affiliation of the signals implies affiliation of the bids. Thus, a rejection of affiliation of the bids means that either the signals are not affiliated or that the bid function is nonmonotonic for reasons other than failure of affiliation. Similarly, although our test is consistent against departures from the null of affiliation of bids and such consistency persists if bid functions are monotonic, it is possible (although unlikely) that nonaffiliated signals combined with a nonmonotonic bid function generate affiliated bids.

Even if the number of bidders n is endogenous or there is unobserved heterogeneity z, our test of affiliation can be applied provided that n and z are affiliated with signals under the null hypothesis, albeit that then bids can be affiliated without signals being affiliated. Our test can also be used in the case of asymmetric bidders, provided that the nature of such asymmetry does not destroy the monotonicity of the bid function.

[1]We will use the word ‘signal’ here to signify either ‘signal’ or ‘value’ depending on the pertaining paradigm.
[2]Except with two bidders, in which case it can be weakened only a little.

A second use of the proposed test is to study the nature of heterogeneity z, which is unobserved to the econometrician. Assume that z is affiliated with bids and that n is exogenous. Then if z is observable to the bidders (independent private values, IPV), as was assumed by Athey et al. (2004), then bids are affiliated with n. However, if z is not observed by the bidders (affiliated private values (APV) or any common values (CV) model), as was allowed for by Krasnokutskaya (2004),[3] then Pinkse and Tan (2005) have shown that bids can be decreasing in n. So a rejection of affiliation of bids and n constitutes a rejection of the IPV model.[4] This conclusion continues to be true when n is endogenous and affiliated with z and bids under the null hypothesis, but a rejection of the null hypothesis (when it is false) is then far less likely.

A further possible application is in the study of collusion. Most tests of collusion that have been proposed (e.g. Bajari and Summers (2002), Bajari and Ye (2003), and Porter and Zona (1993)) exploit the fact that collusion generates asymmetry in the bid distributions.[5] Others use informal arguments (e.g. Marshall and Marx (2008)) or a structural model (Baldwin et al. (1997)) to assert/detect the presence of collusion. We exploit a difference in dependence structure. For instance, if in an IPV auction, ring members other than the designated winner do not submit bids and cartel participation varies across auctions, then bids are no longer necessarily affiliated with n.

Asymmetry in the bid distributions also arises if there is asymmetry in the signal distributions, so finding asymmetry using e.g. the Bajari and Ye test does not 'prove' collusion. Likewise, there are explanations other than collusion that could explain rejection of the competitive null hypothesis using one of the methods we propose. So multiple tests of collusion should be used to form a more complete picture (see Harrington (2008)).

The proposed test is intended as a diagnostic test and hence does not rely on a model. An advantage is that it is widely applicable and can help direct one to the right model. For instance, in our empirical implementation (section 5), some of our test results strongly suggest that in one of the auction data sets used the number of bidders is endogenous. A second plus is that the test results are not colored by the model; a misspecified model generally causes any subsequent (affiliation) test results to be invalid. We favor using tests before and after the formulation of a model, e.g. first

[3] In fact, Krasnokutskaya estimates the fraction of z which is observable to the bidders within a fully structural model.

[4] Please note that a direct test of how bids at a particular signal vary with n requires a correctly specified structural model. If the bid function for a fixed set of signals is decreasing in n, then a test of affiliation will pick this up (in the limit), but a traditional reduced form test (e.g. Gilley and Karels, 1981) consisting of a regression of bids on n will not.

[5] More precisely: exchangeability fails.

on the raw data and subsequently on the signals implied by the model obtained along the lines of Guerre et al. (2000).

Our test is based on EDF's, i.e. standard nonparametric estimates of distribution functions, and we do not require any 'bandwidths,' as would be the case with nonparametric kernel estimation. However, our test necessitates the choice of a sample–size dependent input parameter β_n; in kernel–based tests one would need to choose both bandwidths and β_n. Our simulation results suggest that the results are affected by the choice of β_n, but not to an extent that should be cause for concern.

Despite the popularity of the affiliation assumption in auction models, econometric methods for testing it have not been well developed. The only papers that we are aware of in this regard are de Castro and Paarsch (2008) and Li and Zhang (2008). De Castro and Paarsch formulate their test within the confines of a model, but their test could be applied to raw data, also. They discretize the distributions of interest which provides multinomial likelihoods. Then, they compare the loglikelihoods that are maximized with and without a number of inequality restrictions implied by affiliation. They derived the null distribution of their test, which is nonpivotal and nonstandard. The null distribution of our test is standard normal and we prove that our test is consistent againt all departures from the null of affiliation. Li and Zhang's (2008) test of affiliation is different from ours in that it is both parametric and it is framed in the context of an entry model.

Our paper is organized as follows. In section 2 we introduce our test. The validity of our test depends on a weak technical condition, which is further explained in section 3. We then study the performance of our test in a series of simulation experiments (section 4), followed by an implementation in three important data sets (section 5).

2. A Consistent Nonparametric Test

Consider an independent and identically distributed sequence of random vectors $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$. Our goal is to test for the affiliation of any of this sequence, say ξ.

The definition of affiliation we use is equivalent to that of Milgrom and Weber (1982), lemma 1. Let $a \vee b$ denote the element–wise maximum of a, b and $a \wedge b$ the element–wise minimum, and let $\mathcal{B}(a, \delta)$ be a cube with volume δ and centroid a.

Definition 1. *The elements of a random vector $\xi \in \mathbb{R}^d$ with joint distribution function F are affiliated if for any two vectors $a, b \in \mathbb{R}^d$ and any $\delta > 0$, $Q(a, b; \delta) = p_\delta(a)p_\delta(b) - p_\delta(a \vee b)p_\delta(a \wedge b) \leq 0$, where $p_\delta(a) = P[\xi \in \mathcal{B}(a, \delta)]$.*

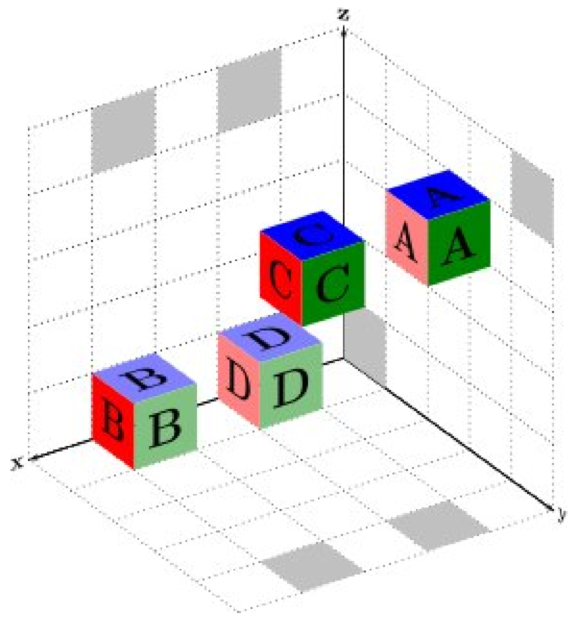

FIGURE 1. Affiliation

The definition is illustrated in figure 1 for $d = 3$. Boxes A,B,C,D correspond to $\mathcal{B}(a, \delta)$, $\mathcal{B}(b, \delta)$, $\mathcal{B}(a \vee b, \delta)$, and $\mathcal{B}(a \wedge b, \delta)$, respectively. Under independence, $Q = 0$ for all a, b, δ. An equivalent definition for continuous F is that $f(a \wedge b)f(a \vee b) \geq f(a)f(b)$ for all a, b, where f is the density function corresponding to F.

Our test statistic is based on the quantity

$$T(Q) = \int m\big\{Q(a, b, \delta)\big\}w(a, b, \delta)dadbd\delta = \int mdW, \tag{1}$$

where $m(Q) = \max(Q, 0)$, w is some continuous nonnegative function such that for any a, b in the support of F, $\lim_{\delta \downarrow 0} w(a, b, \delta) = w^*(a, b) > 0$. By definition $T(Q)$ is positive whenever Q is positive on a set of nonzero measure.[6]

It is possible to estimate T directly, but the limiting distribution of the resulting test statistic would be nonstandard, depend on Q, and would have to be simulated. We have opted instead to adjust the 'kink' in m by the introduction of a sample–size–dependent function g_n. Dropping the (a, b, δ)–argument, let

$$T_n(Q) = \int_{Q > -\beta_n} QdW, \tag{2}$$

[6]It would be possible to develop a Kolmogorov–Smirnov–style test instead of using our Cramér–von Mises approach. We have not done so because of the greater computational difficulty with Kolmogorov–Smirnov type tests.

with $\beta_n \prec 1$ an input parameter, where '$\prec$' indicates that the quantity on the left converges faster than the quantity on the right. We introduce β_n to obtain a desirable limit distribution under the null hypothesis.

We can then estimate the function Q by its sample analog, $\hat{Q}$, given by

$$\hat{Q}(a,b,\delta) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i}^{n} h^*_{ij},$$

with $h^*_{ij} = I\big(\xi_i \in \mathcal{B}(a,\delta)\big) I\big(\xi_j \in \mathcal{B}(b,\delta)\big) - I\big(\xi_i \in \mathcal{B}(a \vee b,\delta)\big) I\big(\xi_j \in \mathcal{B}(a \wedge b,\delta)\big)$. Our test statistic then has the form

$$\hat{\tau} = \sqrt{n} \frac{T_n(\hat{Q})}{2\hat{v}}, \tag{3}$$

where $\hat{v} = \sqrt{\hat{v}^2}$, where $\hat{v}^2$ estimates some positive v^2. The idea is that as the sample size increases, $\hat{Q}$ converges to Q and T_n converges to T, so $T_n(\hat{Q})$ converges to $T(Q)$, which is zero under affiliation and positive absent affiliation. Since our (yet to be defined) $\hat{v}$ is bounded, the $\sqrt{n}$–norming then ensures consistency of our test.

So for consistency all that is required is that $\hat{v}$ is positive and bounded, but its choice is motivated by the test statistic properties under the null. Let $h_{ij} = \int_{Q=0} (h^*_{ij} + h^*_{ji}) dW/2$ and $v^2 = VE[h_{12}|\xi_1] = E[h_{12}h_{13}] - (Eh_{12})^2$. Further, let $h_{nij} = \int_{\hat{Q} > -\beta_n} (h^*_{ij} + h^*_{ji}) dW/2$, and

$$\hat{v}^2 = \big(n(n-1)(n-2)\big)^{-1} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq i,j} h_{nij} h_{nit} - \Big(\big(n(n-1)\big)^{-1} \sum_{i=1}^{n} \sum_{j \neq i}^{n} h_{nij}\Big)^2. \tag{4}$$

We are now in a position to state our consistency theorem.

Theorem I (Consistency)**.** *For any $C < \infty$, $\lim_{n\to\infty} P[\hat{\tau} > C] = 1$.*

The consistency theorem is simple and essentially requires no assumptions. Asymptotic validity is trickier. The main problem is that if the elements of ξ are strictly affiliated, then any test will necessarily be conservative in that the asymptotic rejection probabilities are less than the intended significance levels. A consequence of this is that if on large parts of the support ξ is strongly affiliated and affiliation is violated only in a small area, our test is likely to have little power in samples of moderate size. We make the following assumption.

Assumption A. *For $\xi \sim F$, either (i) the elements of ξ are independent, or (ii) for some $0 < \gamma < \infty$ and some $0 < c_\gamma < 1$, the function $\psi(t) = \int_{-t < Q < -c_\gamma t} dW$ satisfies $\limsup_{t \downarrow 0} \big(t^\gamma / \psi(t)\big) < \infty$.*

Part (ii) of assumption A is nonprimitive and conditions under which it is satisfied are discussed in section 3. It should be noted, however, that generating a relevant scenario under which (i) and (ii) are both violated takes some work. Typically γ can be taken to equal $d/(2d_c)$, where d_c is the number of elements of ξ with continuous distributions. So in case all variables are continuous, $\gamma = 1/2$. The case where all variables are discrete is of little practical interest and the null of affiliation can then be tested more directly by comparing probabilities of each of the realization combinations occurring.

Theorem II (Asymptotic Validity). *Let assumption A hold and let C_α be the $1-\alpha$ quantile of the standard normal distribution. If β_n is chosen such that $1 \succ \beta_n \succ n^{-1/(2+2\gamma)}$ then under the null hypothesis, $\limsup_{n\to\infty} P\big[\hat{\tau} > C_\alpha\big] \leq \alpha$, with equality in case (i) of assumption A*

Note that in many instances our test is conservative in the sense that it rejects less often than the significance level would suggest, even asymptotically. This is natural if one considers the case of a t–test for the mean of an i.i.d. sequence, where the null hypothesis is that the population mean is either zero or negative. If the population mean in fact is negative, then such a t–test will reject less than 5% of the time at a 5% level of significance. Indeed, in the limit the probability of rejection is then zero. Our situation is similar, albeit that now independence is the knife edge case.

3. Assumption A

In this section we investigate conditions under which part (ii) of assumption A is satisfied. The discussion below presumes throughout that the null of affiliation is satisfied, but that the vector ξ does not entirely consist of mutually independent variates. We first state and then explain a sufficient condition for assumption A(ii).

Assumption B. *There exists a set $S \in \mathbb{R}^d \times \mathbb{R}^d$ of positive measure and numbers $0 < \rho, \bar{\delta}, c_\rho, C_\rho < \infty$ such that*

$$\forall (a,b) \in S,\, 0 < \delta < \bar{\delta} : c_\rho \delta^\rho \leq -Q(a,b,\delta) \leq C_\rho \delta^\rho.$$

Theorem III. *If assumption B is satisfied then so is part (ii) of assumption A for $\gamma = 1/\rho$.*

To illustrate assumption B, first suppose that ξ has a continuous density f at $a^*, b^*, a^* \vee b^*, a^* \wedge b^*$ and $\lambda(a^*, b^*) = f(a^* \vee b^*) f(a^* \wedge b^*) - f(a^*) f(b^*) > 0$. Then for some $0 < \delta^*, \underline{\lambda}, \bar{\lambda} < \infty$, $\mathcal{B}(a^*, 2^d\delta^*), \dots, \mathcal{B}(a^* \wedge b^*, 2^d\delta^*)$ do not intersect and for all $a \in \mathcal{B}(a^*, \delta^*), b \in \mathcal{B}(b^*, \delta^*)$ we have $\underline{\lambda} \leq \lambda(a,b) \leq \bar{\lambda}$. Then by construction,

$$\forall \delta \in (0, \delta^*), (a,b) \in S : \underline{\lambda}\delta^2 \leq -Q(a,b,\delta) \leq \bar{\lambda}\delta^2,$$

and assumption B is satisfied with $\rho = 2$.

If ξ contains discrete regressors then the above argument largely carries through, albeit that $|Q|$ only increases at a rate of $\delta^{2d_c/d}$.

4. Experiments

To investigate the performace of our test statistic we have conducted a number of simulation experiments. In our first set of experiments we investigate its behavior under the null hypothesis, and especially how its size varies with the choice of input parameter β_n. Consider figure 2. All

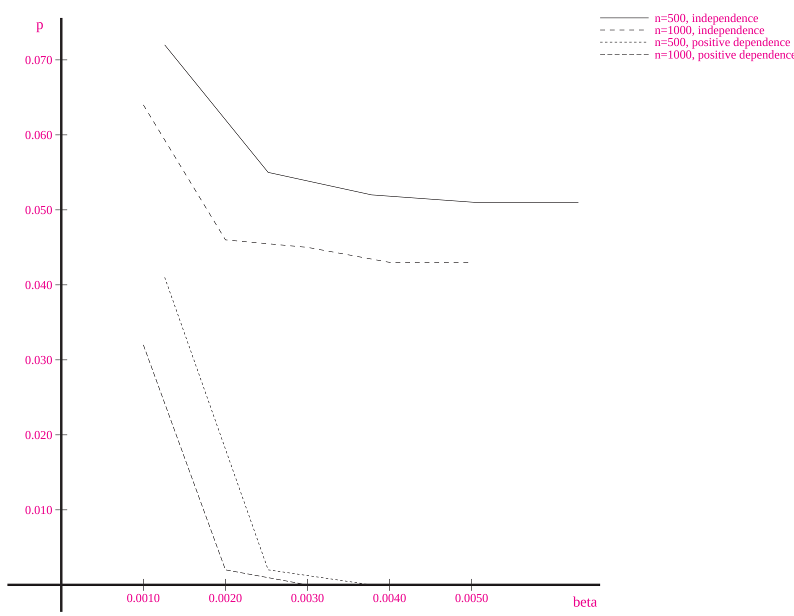

Figure 2. Size and β_n

four curves depict the size of the test as a function of β_n if $d = 2$; results for $d = 3$ are not depicted here, but they are similar. The top two curves are for the case in which the random variates are independent and are drawn from a standard uniform distribution. The size curves are both flat and close to the desired value of 0.05, which is encouraging. As expected, however, with positive dependence our test rejects less than 5% of the time, even in large samples; see the discussion in section 2.

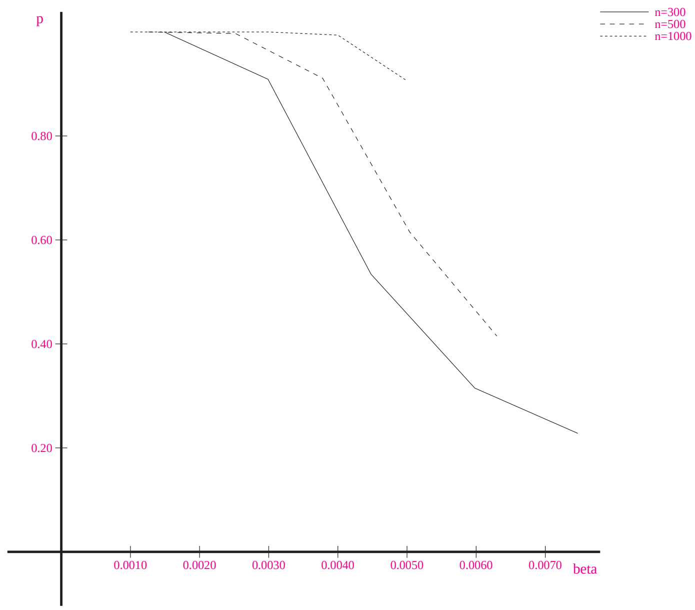

Figure 3. Power and β_n

The choice of β_n also affects the power. The results in figure 3 are for the scenario $\xi_{i1} = (u_{i1} + u_{i2})/2$, $\xi_{i2} = (u_{i1} - u_{i2} + 1)/2$, where u_{i1}, u_{i2} are standard uniform random variables. As one might expect, a smaller choice of β_n results in greater power and more observations means more power.

Given the above results we use $\beta_n = 0.02/\sqrt[3]{n}$ from hereon. We have conducted two more sets of experiments in which the null of affiliation is violated with $d = 2$. The results are tabulated in table 1. The data generating processes (DGPs) used were created to allow for affiliation to hold on part of the support and be violated elsewhere. They are defined as follows.

DGP1 Let u_{i1}, u_{i2} be mean zero unit variance joint normal random variates with correlation ρ. Then ξ_{i1}, ξ_{i2} are given by

$$\begin{cases} \Phi(u_{i1})/2,\ \Phi(u_{i2})/2, & \text{w.p. } 0.25, \\ \text{independent } U(0,0.5),\ U(0,0.5), & \text{w.p. } 0.25, \\ \text{independent } U(0.5,1),\ U(0,0.5), & \text{w.p. } 0.25, \\ \text{independent } U(0.5,1),\ U(0.5,1), & \text{w.p. } 0.25, \end{cases}$$

where Φ is the standard normal distribution function.

DGP2 For a fixed $a \in (0,1)$, let u_{i1}, u_{i2}, u_{i3} be independent random variables from $U(0,1), U(0,1), U(0,a)$, respectively. Then, ξ_{i1} and ξ_{i2} are given by

$$\begin{cases} (u_{i1} + u_{i3})/2,\ (u_{i2} - u_{i3} + a)/2, & \text{if } u_{i1} < a \text{ and } u_{i2} < a \\ (u_{i1} + (1-a)u_{i3}/a + a)/2,\ (u_{i2} + (1-a)u_{i3}/a + a)/2, & \text{if } u_{i1} \geq a \text{ and } u_{i2} \geq a \\ u_{i1},\ u_{i2}, & \text{otherwise.} \end{cases}$$

Given the absence of a natural competitor to our test, our power experiments are of limited use. Nevertheless, it is good to see that our test can pick up even modest violations from the null hypothesis; $\rho = -0.8$ in DGP1 corresponds to a correlation of only -0.049 between ξ_{i1} and ξ_{i2}. DGP2 is designed to illustrate that affiliation is a stronger notion of positive dependence than positive correlation. In particular, when a is chosen to be close to 0, ξ_{i1} and ξ_{i2} are positively correlated but they are not affiliated. As one would expect, the power of the test increases as the amount of negative dependence increases and the number of observations increases. We conducted

		$n = 300$	$n = 500$	$n = 1000$
DGP1	$\rho = -0.6$ (-0.032)	0.115	0.129	0.210
	$\rho = -0.7$ (-0.040)	0.133	0.184	0.353
	$\rho = -0.8$ (-0.049)	0.167	0.254	0.684
DGP2	$a = 0.50$ (-0.001)	1.000	1.000	1.000
	$a = 0.10$ (0.245)	1.000	1.000	1.000
	$a = 0.03$ (0.397)	0.950	0.997	1.000
	$a = 0.01$ (0.463)	0.377	0.674	0.968

The numbers in parentheses show the Monte Carlo correlation between ξ_{i1} and ξ_{i2}.
TABLE 1. Power of the test

similar experiments for the $d = 3$ case, which are not tabulated since they carry largely the same message.

5. Applications

We implemented our test in three data sets, namely the Offshore Continental Shelf (OCS), California Department of Transport (Caltrans), and Russian Federal Subsoil Resources Management Agency (RFSRMA) data sets. In the case of both OCS and RFSRMA the object auctioned is the right to drill for oil (and gas). The Caltrans data set contains data on construction projects.[7]

The OCS data has been used in a number of other papers, including Hendricks and Porter (1988) and Hendricks et al. (2003). The tracts were auctioned off using a first price sealed bid mechanism. Data are available on all bids, the identity of the bidders and any ex post revenue the tracts generated for the winner plus some tract–specific data such as its size and location. We used only data for wildcat tracts in the 1954–1970 period. The number of tracts used is 1,168. Twelve large firms, who collectively win the lion's share of auctions, and a large number of fringe firms participate in these auctions; joint bids are allowed. More details can be found in Hendricks et al. (2003). Although the object sold in the RFSRMA auctions is similar, the data are quite different in nature. They are ascending bid auctions and only the winning bids are recorded. Moreover, there are 117 auctions instead of 1,168 and the government guarantees the presence of a stated minimum amount of oil in the field. Marshall and Marx (2008) provide a more detailed description of this data set and note that certain features of the bid data are suggestive of collusion. The Caltrans data consist of data on bids (including identities and locations of bidders) and a small number of project characteristics for 2,152 construction projects between January 2003 and January 2008, inclusive.

We first used our test to test for affiliation between two randomly chosen (log) bids per auction (OCS and Caltrans), both on the raw data and conditional on ex post revenue and the number of potential bidders[8] (OCS) and engineer's estimate (Caltrans). We then tested for affiliation between a randomly chosen bid and the number of bidders (all three data sets),[9] again with and without conditioning on other variables. We also tested for affiliation between minimum bids and the number of bidders and the minimum of two randomly selected bids and the number of bidders. For the OCS data we conducted all our experiments both including and excluding the fringe bidders.

[7]The OCS data are available from Ken Hendricks, Rob Porter, or the authors. The RFSRMA data set is available from the Center for Auctions, Procurements and Competition Policy website. The Caltrans data are available from the California Department of Transportation website.

[8]See Hendricks et al. (2003).

[9]Since the Caltrans auctions are procurements, we use minus the number of bidders here.

Conditioning affected the values of the test statistics, but the qualitative conclusions are identical across the board. In all but one set of tests we cannot reject the null hypothesis of affiliation. The only exception is the test of affiliation between the lowest bid (including fringe firms) and the number of bidders in the OCS auction case. Violation of the affiliation assumption in that scenario is hardly surprising; even in an IPV auction the minimum bid is often not affiliated with the number of bidders.[10]

The surprising finding is that affiliation could not be rejected in any of the other cases. In fact, the test statistic values were invariably negative and typically well away from zero. One possibility is that the proposed test lacks power. Given the results of our simulation experiments, the number of auctions and the uniformity of the results, we discount this possibility as a primary cause. A second possibility is that the independence assumption across auctions is violated, e.g. because of capacity constraints (e.g. Jofre–Bonet and Pesendorfer, 2003), but this would only affect the denominator of the test statistic and not its sign.

The results are arguably plausible in the Caltrans case since construction procurement auctions are typically modelled as IPV auctions, albeit that cost uncertainties could produce a common value component.[11] But for the drilling rights auctions, and especially the OCS auctions, this finding is less credible.

Mineral rights auctions are generally modelled as common values, with the notable exception of Li et al. (2000) who used an APV framework. Both with CV and APV, however, bids are not affiliated with the (exogenous) number of bidders. By far the most likely explanation for our results is that the number of bidders is endogenous. Endogeneity can arise because higher value tracts attract more bidders, there are entry costs (see e.g. Li (2005)), there is a binding reserve price, there are credit constraints, or (less important in the OCS case) there is collusion. As noted before, for the OCS auctions the results are qualitatively the same after conditioning on ex post revenue, but possibly the tract has geological features of the tract that would generate a lower prediction of the amount of oil present, common to all participants, causing some to drop out because of entry costs. Whatever the precise cause, the conclusion of our results is clearly that modelling the entry decision is important.

[10] E.g. if the value distribution is standard uniform.

[11] In a private conversation, an employee of a large Dutch construction firm which had participated in a large bidding ring in fact complained that firms had identical but uncertain costs, which suggests a common value environment (and is puzzling since such uncertainty increases expected profit).

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Appendix A. Consistency

Lemma A1. $\mathcal{H}^* = \{h^*(\xi,\tilde\xi) = I\big(\xi \in \mathcal{B}(a,\delta)\big)I\big(\tilde\xi \in \mathcal{B}(b,\delta)\big) - I\big(\xi \in \mathcal{B}(a \vee b,\delta)\big)I\big(\tilde\xi \in \mathcal{B}(a \wedge b,\delta)\big) :$ $(a,\delta),\ (b,\delta) \in \mathbb{R}^d \times \mathbb{R}^+\}$ *forms a Euclidean class of functions with an envelop function* $H(\xi,\tilde\xi) = 2$.

Proof. Note first that

$$\mathcal{F} = \{\phi(\xi,\tilde\xi) = I\big(\xi \in \mathcal{B}(a,\delta)\big)I\big(\tilde\xi \in \mathcal{B}(b,\delta)\big) :\ (a,\delta),\ (b,\delta) \in \mathbb{R}^d \times \mathbb{R}^+\}$$

forms a Euclidean class of functions with an envelope function $F(\xi,\tilde\xi) = 1$, because $\{\mathcal{B}(a,\delta)\times\mathcal{B}(b,\delta)\}$ is a collections of cells in $\mathbb{R}^{2d}$, which is a Vapnik–Cervonenkis (VC) class of sets with its VC index bounded by $4d+1$ (see e.g. example 2.6.1 of van der Vaart and Wellner (1996)), and therefore $\mathcal{F}$

forms a Euclidean class of functions by lemma 19 of Nolan and Pollard (1987). Similarly,

$$\mathcal{F}^* = \{\phi^*(\xi, \tilde{\xi}) = -I\big(\xi \in \mathcal{B}(a \vee b, \delta)\big) I\big(\tilde{\xi} \in \mathcal{B}(a \wedge b, \delta)\big) : \ (a, \delta), \ (b, \delta) \in \mathbb{R}^d \times \mathbb{R}^+\}$$

forms a Euclidean class of functions with an envelope function $F^*(\xi, \tilde{\xi}) = 1$. Since $\mathcal{H}^* \subset \mathcal{F} + \mathcal{F}^*$, the lemma follows from corollary 17 of Nolan and Pollard (1987). □

Lemma A2. $\mathcal{H} = \{h(\xi, \tilde{\xi}) = \big(h^*(\xi, \tilde{\xi}) + h^*(\tilde{\xi}, \xi)\big)/2 : \ \ h^*(\cdot, \cdot) \in \mathcal{H}^*\}$ *forms a Euclidean class of functions with an envelope function* $H(\xi, \tilde{\xi}) = 2$.

Proof. It follows from lemma A1 and corollary 17 of Nolan and Pollard (1987). □

Lemma A3. *For each* ξ*, let* $P\mathcal{H}$ *be the class of functions* $Ph(\xi, \cdot)$ *with* $h(\cdot, \cdot) \in \mathcal{H}$*. Then,* $P\mathcal{H}$ *forms a Euclidean class of functions with an envelop function* $PH = 2$.

Proof. It follows from corollary 21 of Nolan and Pollard (1987), because $\mathcal{H}$ is a uniformly bounded Euclidean class of functions. □

Lemma A4. *For some Gaussian process* $\mathcal{G}$*, (i)* $\sqrt{n}(\hat{Q} - Q) \xrightarrow{w} \mathcal{G}$ *and (ii)* $\sup |\hat{Q} - Q| \prec 1$.

Proof. Since $\sqrt{n}(\hat{Q} - Q)$ is a U–process with a function class $\mathcal{H}$, we will follow Nolan and Pollard (1987, 1988) (see also de la Peña and Giné (1999)). We will write $N_p(\epsilon, R, \mathcal{F}, F)$ for the (L_p) ϵ–covering number of a class of functions $\mathcal{F}$ with an envelope function F; i.e. $N_p(\epsilon, R, \mathcal{F}, F)$ is the smallest cardinality for a subclass $\mathcal{F}^*$ of $\mathcal{F}$ such that $\min_{\mathcal{F}^*} R(|\phi - \phi^*|^p) \le \epsilon^p R(F^p)$ for each $\phi \in \mathcal{F}$, where $R(\cdot)$ denotes the integral with respect to the probability measure R. Following Nolan and Pollard (1988), we will also write $J(\delta, R, \mathcal{F}, F)$ for the L_2–covering integral $\int_0^\delta \log N_2(\epsilon, R, \mathcal{F}, F) d\epsilon$.

Since $\mathcal{H}$ is a Euclidean class with an envelope $H = 2$, there exist some constants K_1 and V_1 such that

$$\sup_R N_1\big(\epsilon, R, \mathcal{H}, H\big) \le K_1 \left(\frac{1}{\epsilon}\right)^{V_1}, \quad \text{for } 0 < \epsilon \le 1,$$

where sup is taken over probability measures R. Since $N_1\big(\epsilon, R, \mathcal{H}, H\big)$ is non–increasing in ϵ, it follows that

$$\sup_R N_1\big(\epsilon, R, \mathcal{H}, H\big) \le K_1 \big(\left(\frac{1}{\epsilon}\right)^{V_1} + 1\big)$$

for any $\epsilon > 0$, which implies part (ii) by theorem 7 of Nolan and Pollard (1987).

For part (i), we use theorem 5 of Nolan and Pollard (1988). Note first that there exist some constant K_2, K_2^*, V_2, and V_2^* such that when $0 < \epsilon \leq 1$,

$$\sup_R N_2\big(\epsilon, R, \mathcal{H}, H\big) \leq K_2 \left(\frac{1}{\epsilon^2}\right)^{V_2}$$

$$\sup_R N_2\big(\epsilon, R, P\mathcal{H}, PH\big) \leq K_2^* \left(\frac{1}{\epsilon^2}\right)^{V_2^*}$$

from the fact that both $\mathcal{H}$ and $P\mathcal{H}$ form Euclidean classes. It then follows that

$$\sup_R J(1, R, \mathcal{H}, H) \leq \int_0^1 \log K_2 \left(\frac{1}{\epsilon^2}\right)^{V_2} d\epsilon < \infty$$

$$\sup_R J(1, R, \mathcal{H}, H)^2 \leq \int_0^1 \left(\log K_2 \left(\frac{1}{\epsilon^2}\right)^{V_2}\right)^2 d\epsilon < \infty$$

$$\sup_R J(1, R, P\mathcal{H}, PH)^2 \leq \int_0^1 \left(\log K_2^* \left(\frac{1}{\epsilon^2}\right)^{V_2^*}\right)^2 d\epsilon < \infty.$$

Lastly, note that as $\delta \downarrow 0$,

$$\sup_R J(\delta, R, P\mathcal{H}, PH) \leq \int_0^\delta \log K_2^* \Big(\left(\frac{1}{\epsilon^2}\right)^{V_2^*}\Big) d\epsilon = \delta \log K_2^* - 2V_2^* \delta \log \delta \to 0.$$

Therefore, the conditions of theorem 5 of Nolan and Pollard (1988) are all satisfied, and part (i) of the lemma follows from it. □

Lemma A5. $T_n(\hat{Q}) - T(Q) \prec 1$.

Proof.

$$T_n(\hat{Q}) - T(Q) = \int_{\hat{Q} > -\beta_n} \hat{Q} dW - \int_{Q>0} Q dW = \int_{\hat{Q} > -\beta_n} (\hat{Q} - Q) dW + \int_{\hat{Q} > -\beta_n, Q \leq -2\beta_n} Q dW$$
$$+ \int_{\hat{Q} > -\beta_n, 0 > Q > -2\beta_n} Q dW - \int_{\hat{Q} < -\beta_n, Q \geq 0} Q dW. \quad (5)$$

RHS1, RHS2 and RHS4 in (5) vanish by lemma A4. Finally, RHS3 is bounded above by zero and below by $-2\beta_n \prec 1$. □

Proof of Theorem I. Since h_{ij} is uniformly bounded, so is $\hat{v}$. Let C_v be an upper bound to $\hat{v}$, let $\mathcal{E}$ be the event that $|T_n(\hat{Q}) - T(Q)| \leq T(Q)/2$, $\mathcal{E}^c$ its complement, and write

$$P[\hat{\tau} \leq C] \leq P\big[\hat{\tau} \leq C, \mathcal{E}\big] + P\big[\mathcal{E}^c\big]. \quad (6)$$

Since $T(Q) > 0$, RHS2 in (6) vanishes by lemma A5. RHS1 in (6) is

$$P\left[\sqrt{n}\frac{T_n(\hat{Q})}{\hat{v}} \leq C, \mathcal{E}\right] = P\left[\sqrt{n}\frac{T_n(\hat{Q}) - T(Q)}{\hat{v}} + \sqrt{n}\frac{T(Q)}{\hat{v}} \leq C, \mathcal{E}\right] \leq P\left[\sqrt{n}\frac{T(Q)}{2C_v} \leq C\right] \prec 1. \quad □$$

Appendix B. Validity

Let $R_n = \{(a,b,\delta) : 0 > Q(a,b,\delta) > -\beta_n/2\}$. Let $Z_n = \int_{Q=0}(\hat{Q}-Q)dW$, $N_n = \int_{R_n}(\hat{Q}-Q)dW$, $K_n = \int_{R_n} QdW$ and $\mathcal{I}_n = \int_{Q\le-\beta_n/2,-\beta_n<\hat{Q}\le 0}\hat{Q}dW$.

Lemma B1. $\limsup_{n\to\infty} P[T_n(\hat{Q}) \neq Z_n + K_n + \mathcal{I}_n] = 0$.

Proof. Note that

$$
\begin{aligned}
T_n(\hat{Q}) &= \int_{Q>-\beta_n/2,\hat{Q}>-\beta_n}\hat{Q}dW + \int_{Q\le-\beta_n/2,\hat{Q}>-\beta_n}\hat{Q}dW \\
&= \int_{Q>-\beta_n/2}\hat{Q}dW - \int_{Q>-\beta_n/2,\hat{Q}\le-\beta_n}\hat{Q}dW + \mathcal{I}_n + \int_{Q\le-\beta_n/2,\hat{Q}>0}\hat{Q}dW \\
&= Z_n + N_n + K_n + I_n - \int_{R_n,\hat{Q}\le-\beta_n}\hat{Q}dW + \int_{Q\le-\beta_n/2,\hat{Q}>0}\hat{Q}dW. \qquad (7)
\end{aligned}
$$

The last two terms are zero with probability approaching one by lemma A4. □

B.1. Under condition (i) of assumption A.

Lemma B2. $\sqrt{n}T_n(\hat{Q}) \xrightarrow{d} N(0,4v^2)$.

Proof. Since $N_n = K_n = I_n = 0$, it suffices to show that $\sqrt{n}Z_n \xrightarrow{d} N(0,4v^2)$. Now, $\sqrt{n}Z_n = n^{-3/2}\sum_{i=1}^n\sum_{j\neq i} h_{ij}$, which is a standard nondegenerate U–statistic. Apply standard U–statistic theory (e.g. theorem A in section 5.5 of Serfling (1980)). □

Lemma B3. $\limsup_{n\to\infty} P\big[\max_{i\neq j}|h_{nij}-h_{ij}|>0\big] = 0$.

Proof. Noting that $h_{nij}-h_{ij} = \int_{\hat{Q}\le-\beta_n,Q=0}(h^*_{ij}+h^*_{ji})dW/2$, it follows that

$$
P\big[\max_{i\neq j}|h_{nij}-h_{ij}|>0\big] \le n^2P\big[|h_{n12}-h_{12}|>0\big] \le n^2P\big[\sup|\hat{Q}-Q|>\beta_n\big] \prec 1,
$$

by lemma A4. □

Lemma B4. $\hat{v}^2 - v^2 \prec 1$.

Proof. Let $\tilde{v}^2 = \big(n(n-1)(n-2)\big)^{-1}\sum_{i=1}^n\sum_{j\neq i}\sum_{t\neq i,j} h_{ij}h_{it}$. Then $\hat{v}^2-\tilde{v}^2 \prec 1$ by lemma B3. Since $\tilde{v}^2$ is an asymmetric U–statistic, one can apply theorem A in section 5.5 of Serfling (1980). □

Lemma B5. $\hat{\tau} \xrightarrow{d} N(0,1)$.

Proof. Combine lemmas B2 and B4 and apply Cramér's theorem to obtain the stated result. □

B.2. **Under condition (ii) of assumption A.**

Lemma B6. $|K_n| \succeq \beta_n^{1+\gamma}$.

Proof. By assumption A(ii),

$$|K_n| = -\int_{Q>-\beta_n/2} QdW \geq -\int_{-\beta_n/2<Q<-c_\gamma\beta_n/2} QdW \geq c_\gamma\beta_n\psi(\beta_n/2)/2 \succeq \beta_n^{1+\gamma}. \qquad \square$$

Lemma B7. $\limsup_{n\to\infty} P[T_n(\hat{Q}) > 0] = 0$.

Proof. By lemma A4, $N_n + Z_n \preceq n^{-1/2}$ and by lemma B6, $K_n \succeq \beta_n^{1+\gamma} \succ n^{-1/2}$. Since $\mathcal{I}_n$ is negative, the stated result follows from lemma B1. $\square$

Proof of Theorem II. In case of condition (i), apply lemma B5, otherwise apply lemma B7. $\square$

Appendix C. Assumption A

Proof of Theorem III. Note that for $c_\gamma = c_\rho/(2^\rho C_\rho)$ and any $(a,b) \in S$, if $t_\rho = (t/C_\rho)^{1/\rho}$, then for sufficiently small $t > 0$,

$$\frac{t_\rho}{2} < \delta < t_\rho \Rightarrow \frac{t}{2^\rho C_\rho} < \delta^\rho < \frac{t}{C_\rho} \Rightarrow \frac{c_\gamma t}{c_\rho} < \delta^\rho < \frac{t}{C_\rho} \Rightarrow C_\rho\delta^\rho < t,\ c_\rho\delta^\rho > c_\gamma t \Rightarrow c_\gamma t < -Q < t.$$

So $\psi(t) = \int_{c_\gamma t<-Q<t} dW \geq \int_{(a,b)\in S;\, t_\rho/2<\delta<t_\rho} dW \succeq t^{1/\rho} = t^\gamma$. $\square$

Lemma C1. *By continuity, (a^*, b^*) is an interior point of a set S^* for which for some $0 < \delta^*, \bar{Q}_f, \underline{Q}_f < \infty$,*

$$\forall (a,b) \in S^*,\ 0 < \delta < \delta^* : \delta\underline{Q}_f \leq |Q| \leq \delta\bar{Q}_f.$$

Let n be large enough to ensure that $\beta_n/\underline{Q}_f < \delta^*$. Then

$$|K_n| \geq \int_{S^*}\int_{0<\delta<\beta_n/\bar{Q}_f} |Q|dW \geq \beta_n\frac{\underline{Q}_f}{\bar{Q}_f}\int_{S^*}\int_{0<\delta<\beta_n/\bar{Q}_f} dW \succeq \beta_n^2.$$